Dispersion of passive tracers in closed basins: beyond the diffusion coefficient

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Abstract

We investigate the spreading of passive tracers in closed basins. If the characteristic length scale of the Eulerian velocities is not very small compared with the size of the basin the usual diffusion coefficient does not give any relevant information about the mechanism of spreading.

We introduce a finite size characteristic time $\tau(\delta)$ which describes the diffusive process at scale δ. When δ is small compared with the typical length

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of the velocity field one has $\tau(\delta) \sim \lambda^{-1}$, where λ is the maximum Lyapunov exponent of the Lagrangian motion. At large δ the behavior of $\tau(\delta)$ depends on the details of the system, in particular the presence of boundaries, and in this limit we have found a universal behavior for a large class of system under rather general hypothesis.

The method of working at fixed scale δ makes more physical sense than the traditional way of looking at the relative diffusion at fixed delay times. This technique is displayed in a series of numerical experiments in simple flows.

I. INTRODUCTION

The understanding of diffusion and transport of passive tracers in a given velocity field has both theoretical and practical relevance in many fields of science and engineering, e.g. mass and heat transport in geophysical flows (for a review see [1,2]), combustion and chemical engineering [3].

One common interest is the study of the mechanisms which lead to transport enhancement as a fluid is driven farther from the motionless state. This is related to the fact that the Lagrangian motion of individual tracers can be rather complex even in simple laminar flows [4,5].

The dispersion of passive scalars in a given velocity field is the result, usually highly non-trivial, of two different contributions: molecular diffusion and advection. In particular, one can have rather fast transport, even without molecular diffusion, in presence of *Lagrangian chaos* , which is the sensitivity to initial conditions of Lagrangian trajectories. In addition, also for a 2D stationary velocity field, where one cannot have Lagrangian chaos [6], in presence of a particular geometry of the streamlines the diffusion can be much larger than the one due only to the molecular contribution, as in the case of spatially periodic stationary flows [7,8].

Taking into account the molecular diffusion, the motion of a test particle (the tracer) is described by the following Langevin equation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) + \boldsymbol{\eta}(t), \tag{1}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian incompressible velocity field at the point $\mathbf{x}$ and time t, $\boldsymbol{\eta}(t)$ is a Gaussian white noise with zero mean and

$$< \eta_i(t)\eta_j(t^{'}) >= 2D_0\delta_{ij}\delta(t - t^{'}), \tag{2}$$

where D_0 is the (bare) molecular diffusivity.

Denoting $\Theta(\mathbf{x}, t)$ the concentration of tracers, one has:

$$\partial_t \Theta + (\mathbf{u} \cdot \boldsymbol{\nabla}) \Theta = D_0 \Delta \Theta. \tag{3}$$

For an Eulerian velocity field periodic in space, or anyway defined in infinite domains, the long-time, large-distance behavior of the diffusion process is described by the effective diffusion tensor D_{ij}^E (*eddy-diffusivity tensor*):

$$D_{ij}^E = \lim_{t \to \infty} \frac{1}{2t} < (x_i(t) - < x_i >)(x_j(t) - < x_j >) >, \tag{4}$$

where now $\mathbf{x}(t)$ is the position of the the tracer at time t, $i, j = 1, \cdots, d$ (being d the spatial dimension) , and the average is taken over the initial positions or, equivalently, over an ensemble of test particles. The tensor D_{ij}^E gives the long-time, large-distance equation for $< \Theta >$ i.e. the concentration field locally averaged over a volume of linear distance much larger than the typical length l_u of the velocity field, according to

$$\partial_t < \Theta > = \sum_{i,j=1}^{d} D_{ij}^E \frac{\partial^2}{\partial x_i \partial x_j} < \Theta > . \tag{5}$$

The above case, with finite D_{ij}^E, is the typical situation where the diffusion, for very large times, is a standard diffusion process. However there are also cases showing the so-called *anomalous diffusion*: the spreading of the particles does not behave linearly with time but has a power law $t^{2\nu}$ with $\nu \neq 1/2$. Transport anomalies are, in general, indicators of the presence of strong correlation in the dynamics, even at large time and space scales [9].

In the case of infinite spatial domains and periodic Eulerian fields the powerful multiscale technique (also known as homogenization in mathematical literature) gives a useful tool for studying standard diffusion and, with some precautions, also the anomalous situations [10].

On the other hand we have to stress the fact that diffusivity tensor (4) is mathematically well defined only in the limit of infinite times, therefore it gives a sensible result only if the characteristic length l_u of the velocity field is much smaller than the size L of the domain.

The case when l_u and L are not well separated is rather common in many geophysical problems, e.g. spreading of pollutants in Mediterranean or Baltic sea, and also in plasma physics. Therefore it is important to introduce some other characterizations of the diffusion

properties which can be used also in non ideal cases. For instance, [11] propose to employ exit times for the study of transport in basins with complicated geometry.

In Section II we introduce a characterization of the diffusion behavior in terms of the typical time $\tau(\delta)$ at scale δ; this allows us to define a finite size diffusion coefficient $D(\delta) \sim \delta^2/\tau(\delta)$. ¿From the shape of $\tau(\delta)$ as a function of δ, one can distinguish different spreading regimes.

In Section III we present the results of numerical experiments in closed basins and present new results relative to the behavior of the diffusion coefficient near the boundary (a detailed discussion is in the appendix).

In Section IV we summarize our results and present conclusions and we discuss the possibility of treatment of experimental data according to the method introduced in Section II.

II. FINITE SIZE DIFFUSION COEFFICIENT

Before a general discussion let us start with a simple example. Consider the relative diffusion of a cloud of N test particles in a smooth, spatially periodic velocity field with characteristic length l_u. We assume that the Lagrangian motion is chaotic i.e. the maximum Lyapunov exponent λ is positive. Denoting with $R^2(t)$ the square of the typical radius of the cloud

$$R^2(t) = \ll |\mathbf{x}_i(t) - \ll \mathbf{x}_i(t) \gg |^2 \gg, \tag{6}$$

where

$$\ll \mathbf{x}_i(t) \gg = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i(t) \tag{7}$$

we expect the following regimes to hold

$$\overline{R^2(t)} \simeq \begin{cases} R^2(0)\exp(L(2)t) & \text{if } \overline{R^2(t)}^{1/2} \ll l_u \\ 2Dt & \text{if } \overline{R^2(t)}^{1/2} \gg l_u \end{cases}, \tag{8}$$

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where $L(2) \geq 2\lambda$ is the generalized Lyapunov exponent [12,13], D is the diffusion coefficient and the overbar denotes the average over initial conditions.

In this paper we prefer to study the relative diffusion (6) instead of the usual absolute diffusion. For spatially infinite cases, without mean drift there is no difference; for closed basins the relative dispersion is, for many aspects, more interesting than the absolute one and, in addition, the latter is dominated by the sweeping induced by large scale flow.

Furthermore we underline that although the dynamics of the ocean circulation is dominated by large mesoscale gyres, the smaller scales activities within the gyres control important local phenomena as deep water formation in North Atlantic and in Mediterranean basin [14]. Therefore the study of relative diffusion could be relevant to describe this small-scale motion and can give crucial informations on the way to parameterize the subgrid scales in ocean numerical global model [15].

Another, at first sight rather artificial, way to describe the above behavior is by introducing the "doubling time" $\tau(\delta)$ at scale δ as follows: we define a series of thresholds $\delta^{(n)} = r^n \delta^{(0)}$, where $\delta^{(0)}$ is the initial size of the cloud, defined according to (6), and then we measure the time $T(\delta^{(0)})$ it takes for the growth from $\delta^{(0)}$ to $\delta^{(1)} = r\delta^{(0)}$, and so on for $T(\delta^{(1)})$, $T(\delta^{(2)})$, ... up to the largest scale under consideration. For the threshold rate r any value can be chosen but too large ones might not separate different scale contributions, though strictly speaking the term "doubling time" refers to the threshold rate $r = 2$.

Performing $\mathcal{N} \gg 1$ experiments with different initial conditions for the cloud, we define the typical doubling time $\tau(\delta)$ at scale δ as

$$\tau(\delta) =< T(\delta) >_e = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} T_i(\delta) \,. \tag{9}$$

Let us stress the fact that the average in (9) is different from the usual time average.

From the average doubling time we can define the finite size Lagrangian Lyapunov exponent as

$$\lambda(\delta) = \frac{\ln r}{\tau(\delta)} \,, \tag{10}$$

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which is a measure of the average rate of separation of two particles at a distance δ. Let us remark that $\lambda(\delta)$ is independent of r, for $r \to 1^+$. For very small separations (i.e. $\delta \ll l_u$) one recovers the standard Lagrangian Lyapunov exponent λ,

$$\lambda = \lim_{\delta \to 0} \frac{1}{\tau(\delta)} \ln r . \tag{11}$$

See [16] for a detailed discussion about these points. In this framework the finite size diffusion coefficient $D(\delta)$ dimensionally turns out to be

$$D(\delta) = \delta^2 \lambda(\delta) . \tag{12}$$

Note the absence of the factor 2, as one can expect by the definition (4), in the denominator of $D(\delta)$ in equation (12); this is due to the fact that $\tau(\delta)$ is a difference of times. For a standard diffusion process $D(\delta)$ approaches the diffusion coefficient D (see eq. (8)) in the limit of very large separations ($\delta \gg l_u$). This result stems from the scaling of the doubling times $\tau(\delta) \sim \delta^2$ for normal diffusion.

Thus the finite size Lagrangian Lyapunov exponent $\lambda(\delta)$, or its counterpart $D(\delta)$, embody the asymptotic behaviors

$$\lambda(\delta) \sim \begin{cases} \lambda & \text{if } \delta \ll l_u \\ D/\delta^2 & \text{if } \delta \gg l_u \end{cases} , \tag{13}$$

One could naively conclude, matching the behaviors at $\delta \sim l_u$, that $D \sim \lambda l_u^2$. This is not always true, since one can have a rather large range for the crossover due to the fact that nontrivial correlations can be present in the Lagrangian dynamics [17].

Another case where the behavior of $\tau(\delta)$ as a function of δ is essentially well understood is 3D fully developed turbulence. For sake of simplicity we neglect intermittency effects. There are then three different ranges:

1. $\delta \ll \eta =$ Kolmogorov length : $1/\tau(\delta) \sim \lambda$;

2. $\eta \ll \delta \ll l =$ typical size of the energy containing eddies: from the Richardson law $\overline{R^2(t)} \sim t^3$ one has $1/\tau(\delta) \sim \delta^{-2/3}$;

3. $\delta \gg l$: usual diffusion behavior $1/\tau(\delta) \sim \delta^{-2}$.

One might wonder that the proposal to introduce the time $\tau(\delta)$ is just another way to look at $\overline{R^2(t)}$ as a function of t. This is true only in limiting cases, when the different characteristic lengths are well separated and intermittency is weak. In [18–20] rather close techniques are used for the computation of the diffusion coefficient in nontrivial cases.

The method of working at fixed scale δ, allows us to extract the physical information at that spatial scale avoiding unpleasant troubles of the method of working at a fixed delay time t. For instance, if one has a strong intermittency, and this is a rather usual situation, $R^2(t)$ as a function of t can appear very different in each realization. Typically one can have, see figure 1a, different exponential rates of growth for different realizations, producing a rather odd behavior of the average $\overline{R^2(t)}$ without any physical meaning. For instance in figure 1b we show the average $\overline{R^2(t)}$ versus time t; at large times we recover the diffusive behavior but at intermediate times appears an apparent anomalous regime which is only due to the superposition of exponential and diffusive contributions by different samples at the same time. On the other hand exploiting the tool of doubling times one has an unambiguous result (see figure 1c).

Of course the interesting situations are those where the different characteristic lengths $(\eta, \; l, \; L)$ are not very different and therefore each scaling regime for $\overline{R^2(t)}$ is not well evident.

III. NUMERICAL RESULTS

Here we present some numerical experiments in simple models with Lagrangian chaos in the zero molecular diffusion limit. Before showing the results, we describe the numerical method adopted.

We choose a passive tracers trajectory having a chaotic behavior, i.e. with a positive maximum Lyapunov exponent, computed by using standard algorithms [21]. Then we place $N-1$ passive tracers around the first one in a cloud of initial size

$$R(0) = \delta(0) = \delta^{(0)} \, ,$$

with $R(0)$ defined by equation (6). In order to have average properties we repeat this procedure reconstructing the passive cloud around the last position reached by the reference chaotic tracer in the previous expansion. This ensures that the initial expansion of the cloud is exponential in time, with typical exponential rate equal to the Lyapunov exponent.

Further we define a series of thresholds $\delta^{(n)} = r^n \delta^{(0)}$ (as described in Section 2) $n = 1, \cdots, n_{max}$ and we measure the time T_n spent in expanding from $\delta^{(n)}$ to $\delta^{(n+1)}$. The value of n_{max} has to be chosen in such a way that $\delta^{(n_{max})} \sim \delta_{max}$, where δ_{max} corresponds to the uniform distribution of the tracers in the basin (see forthcoming discussion and the Appendix). Each realization stops when $\delta(t) = \delta^{(n_{max})}$.

Therefore following [16] we define a scale dependent Lagrangian Lyapunov exponent as:

$$\lambda(\delta^{(n)}) = \frac{1}{<T_n>_e} \ln r = \frac{1}{\tau(\delta^{(n)})} \ln r. \tag{14}$$

In equation (14) we have implicitly assumed that the evolution of the size $\delta(t)$ of the cloud is continuous in time. This is not true in the case of discontinuous processes such as maps or in the analysis of experimental data taken at fixed delay times. Denoting T_n the time to reach size $\tilde{\delta} \geq \delta^{(n+1)}$ from $\delta^{(n)}$, now $\tilde{\delta}$ is a fluctuating quantity, equation (14) has to be modified as follows [16]:

$$\lambda(\delta^{(n)}) = \frac{1}{<T_n>_e} \left\langle \ln \left(\frac{\tilde{\delta}}{\delta^{(n)}} \right) \right\rangle_e . \tag{15}$$

In our numerical experiments we have the regimes described in sect. 2: exponential regime , i.e. $\lambda(\delta) = \lambda$, and diffusion-like regime i.e. $\lambda(\delta) = D/\delta^2$, at least if the size L of the basin is large enough.

For cloud sizes close to the saturation value δ_{max} we expect the following behavior to hold for a broad class of systems:

$$\lambda(\delta) = \frac{D(\delta)}{\delta^2} \propto \frac{(\delta_{max} - \delta)}{\delta} . \tag{16}$$

The constant of proportionality is given by the second eigenvalue of the Perron-Frobenius operator which is related to the typical time of exponential relaxation of tracers' density to the uniform distribution Actually, the analytical evaluation of this eigenvalue can be performed only for extremely simple dynamical systems (for instance random walkers, as shown in the Appendix). As a consequence the range of validity for (16) can be assessed only by numerical simulation.

A. A model for transport in Rayleigh-Bénard convection

The advection in two dimensional incompressible flows is described, in absence of molecular diffusion, by Hamiltonian equation of motion where the Hamilton function is the stream function ψ:

$$\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}. \tag{17}$$

If ψ is time-dependent the system (17) is non-autonomous and in general non-integrable, then chaotic trajectories may exist.

One example is the model introduced in [22] to describe the chaotic advection in the time-periodic Rayleigh-Bénard convection. It is defined by the stream function:

$$\psi(x, y, t) = \frac{A}{k} \sin \left\{ k \left[x + B \sin(\omega t) \right] \right\} W(y), \tag{18}$$

where $W(y)$ is a function that satisfies rigid boundary conditions on the surfaces $y = 0$ and $y = a$ (we use $W(y) = \sin(\pi y/a)$). The direction y is identified with the vertical direction and the two surfaces $y = a$ and $y = 0$ are the top and bottom surfaces of the convection cell. The time dependent term $B \sin(\omega t)$ represents lateral oscillations of the roll pattern which mimic the even oscillatory instability [22].

Trajectories starting near the roll separatrices could have positive Lyapunov exponent and thus display chaotic motion and diffusion in the x direction. It is remarkable that in spite of the simplicity of the model, the agreement of the numerical results with experimental ones is quite good [22].

Defining a passive cloud in the x direction (i.e. a segment) and performing the expansion experiment described in the previous section we have that, until δ is below a fraction of the dimension of the cell, $\lambda(\delta) = \lambda$ (figure 2a). For larger values of δ we have the standard diffusion $\lambda(\delta) = D/\delta^2$ with good quantitative agreement with the value of the diffusion coefficient evaluated by the standard technique, i.e. using $\overline{R^2(t)}$ as a function of time t (compare figure 2a with figure 2b).

To confine the motion of tracers in a closed domain, i.e. $x \in [-L, L]$, we must slightly modify the streamfunction (18). We have modulated the oscillating term in such a way that for $|x| = L$ the amplitude of the oscillation is zero, i.e. $B \rightarrow B \sin(\pi x/L)$ with $L = 2\pi n/k$ (n is the number of convective cells). In this way the motion is confined in $[-L, L]$.

In figure 3 we show $\lambda(\delta)$ for two values of L. If L is large enough one can well see the three regimes, the exponential one, the diffusive one and the saturation given by equation (16). Decreasing L the range of the diffusive regime decreases, and for small values of L it disappears.

B. Modified Standard Map

One of the simplest deterministic dynamical system displaying both exponential growth of separation for close trajectories and asymptotic diffusive behavior is the standard (Chirikov - Taylor) mapping [23]. It is customarily defined as

$$
\begin{cases}
x_{n+1} = x_n + K \sin y_n \\
y_{n+1} = y_n + x_{n+1} \qquad \mathrm{mod}\ 2\pi
\end{cases}
\tag{19}
$$

This mapping conserves the area in the phase space. It is widely known that for large enough values of the nonlinearity strength parameter $K \gg K_c \simeq 1$ the motion is strongly chaotic in almost all the phase space. In this case the standard map, in the x-direction mimics the behavior of a one-dimensional random walker, still being deterministic, and so one expects the behavior of $\lambda(\delta)$ to be quite similar to the one already encountered in the model for Rayleigh-Bénard convection without boundaries. Numerical iteration of (19) for a cloud of

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particles clearly shows the two regimes described in (13), similar to that showed for the model discussed in the previous section.

We turn now to the more interesting case in which the domain is limited by boundaries reflecting back the particle. To achieve the confinement of the trajectory inside a bounded region we modify the standard map in the following way

$$
\begin{cases}
x_{n+1} = x_n + Kf(x_{n+1})\sin y_n \\
y_{n+1} = y_n + x_{n+1} - Kf'(x_{n+1})\cos y_n \mod 2\pi.
\end{cases}
\tag{20}
$$

where $f(x)$ is a function which has its only zeros in $\pm L$. Since the mapping is defined in implicit form, the shape of f must be chosen in such a way to assure a unique definition for (x_{n+1}, y_{n+1}) given (x_n, y_n). For any f fulfilling this request the mapping (20) conserves the area. A trial choice could be

$$
f(x) =
\begin{cases}
1 & |x| < \ell \\
\dfrac{L - |x|}{L - \ell} & \ell < |x| < L
\end{cases}
\tag{21}
$$

Strictly speaking this is not quite an appropriate choice, since it renders the map discontinuous at $|x| = \ell$, but this is an irrelevant point and it is easy to bypass this obstacle by assuming a suitably smoothed version of (21).

Performing the doubling times computation (9) one recovers both the exponential and diffusive regimes for $\lambda(\delta)$, and in addition one has the saturation regime (16). Figure 4 shows the behavior of the scale dependent diffusion coefficient $D(\delta)$ (12). Approaching the saturation value δ_{max} the diffusion coefficient quickly drops to zero, following the asymptotic law (16) derived in the appendix. The qualitative behaviors in figure 4 do not depend on the details of the function f.

C. Point vortices in a Disk

As another example, we consider the two-dimensional time-dependent flow generated by the motion of N point vortices in a closed domain [24]. For a unitary disk the positions

of the vortices ($x_i = r_i \cos\theta_i, y_i = r_i \sin\theta_i$), with circulation Γ_i, evolve according to the Hamiltonian dynamics

$$\dot{x}_i = \frac{1}{\Gamma_i}\frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{1}{\Gamma_i}\frac{\partial H}{\partial x_i} \tag{22}$$

where the Hamiltonian is

$$H = -\frac{1}{4\pi}\sum_{i>j}\Gamma_i\Gamma_j \log\left[\frac{r_i^2 + r_j^2 - 2r_ir_j\cos(\theta_i - \theta_j)}{1 + r_i^2 r_j^2 - 2r_ir_j\cos(\theta_i - \theta_j)}\right] + \frac{1}{4\pi}\sum_{i=1}^{N}\Gamma_i^2\log(1 - r_i^2) \tag{23}$$

Passive tracers evolve according to (17) with ψ given by

$$\psi(x,y) = -\frac{1}{4\pi}\sum_{i}^{N}\Gamma_i\log\left[\frac{r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)}{1 + r^2 r_i^2 - 2rr_i\cos(\theta - \theta_i)}\right] \tag{24}$$

where ($x = r\cos\theta, y = r\sin\theta$) denote the tracer position.

Figure 5 shows the relative diffusion as a function of time in a system with 4 vortices. Apparently there is an intermediate regime of anomalous diffusion. On the other hand from figure 6 one can see rather clearly that, with the method of working at fixed scale, only two regimes survive: the exponential one and that one due to the saturation. Comparing figure 5 and figure 6 one understands that the mechanism described in Section 2 has to be held for responsible of this spurious anomalous diffusion. We stress the fact that these misleading behaviors are due to the superposition of different regimes and that the method of working at fixed scale has the advantage to eliminate this trouble.

The absence of the diffusive range $\lambda(\delta) \sim \delta^{-2}$ is due to the fact that the characteristic length of the velocity field, which is comparable with the typical distance between two close vortices, is not much smaller than the size of the basin.

IV. CONCLUSIONS

In this paper we investigated the relative dispersion of passive tracers in closed basins. Instead of the customary approach based on the average size of the cloud of tracers as a function of time, we introduced a typical inverse time $\lambda(\delta)$ which characterizes the diffusive process at fixed scale δ.

For very small values of δ, $\lambda(\delta)$ coincides with the maximum Lagrangian Lyapunov exponent which is positive in the case of chaotic Lagrangian motion. For larger δ the shape of $\lambda(\delta)$ depends on the detailed mechanism of spreading which is given by the structure of the advecting flow, which is in turn conditioned by the presence of boundaries. In the case of diffusive regime, one expects the scaling $\lambda(\delta) \simeq \delta^{-2}$, which leads to a natural generalization of the diffusion coefficient as $D(\delta) = \lambda(\delta)\delta^2$.

The effectiveness of finite size quantities $\lambda(\delta)$ or $D(\delta)$ in characterizing the dispersion properties of a cloud of particles is demonstrated by several numerical examples.

Furthermore, when δ gets close to its saturation value (i.e. the characteristic size of the basin), a simple argument gives the shape of $\lambda(\delta)$ which is expected to be universal with respect to a wide class of dynamical systems.

In the limiting case when the characteristic length of the Eulerian velocity l_u and the size of the basin L are well separated, the customary approach and the proposed method give the same information. In presence of strongly intermittent Lagrangian motion, or when l_u/L is not much smaller than one, the traditional method can give misleading results, for instance apparent anomalous scaling over a rather wide time interval, as demonstrated by a simple example.

We want to stress out that our method is very powerful in separating the different scales acting on diffusion and consequently it could give improvement about the parameterization of small-scale motions of complex flows. The proposed method could be also relevant in the analysis of drifter experimental data or in numerical models for Lagrangian transport, in particular for addressing the question about the existence of low dimensional chaotic flows.

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APPENDIX

In this appendix we present the derivation of the asymptotic behavior (16) of $\lambda(\delta)$ for δ near to the saturation, for a one dimensional Brownian motion in the domain $[-L, L]$, with reflecting boundary conditions. The evolution of the probability density p is ruled by the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}D\frac{\partial^2 p}{\partial x^2} \tag{25}$$

with the Neumann boundary conditions

$$\frac{\partial p}{\partial x}(\pm L) = 0. \tag{26}$$

The general solution of (25) is

$$p(x,t) = \sum_{k=-\infty}^{\infty} \hat{p}(k,0)e^{ikx}e^{-t/\tau_k} + c.c \tag{27}$$

where

$$\tau_k = \left(\frac{D}{2}\frac{\pi^2}{L^2}k^2\right)^{-1}, \quad k = 0, \pm 1, \pm 2, ... \tag{28}$$

At large times p approaches the uniform solution $p_0 = 1/2L$. Writing p as $p(x,t) = p_0 + \delta p(x,t)$ we have, for $t \gg \tau_1$,

$$\delta p \sim \exp(-t/\tau_1). \tag{29}$$

The asymptotic behavior for the relative dispersion $R^2(t)$ is

$$R^2(t) = \frac{1}{2}\int (x - x')^2 p(x,t)p(x',t) \, dx \, dx' \tag{30}$$

For $t \gg \tau_1$ using (29) we obtain

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$$R^2(t) \sim \left(\frac{L^2}{3} - A e^{-t/\tau_1} \right). \tag{31}$$

Therefore for $\delta(t) = R(t)$ one has

$$\delta(t) \sim \left(\frac{L}{\sqrt{3}} - \frac{\sqrt{3}A}{2L} e^{-t/\tau_1} \right) \tag{32}$$

The saturation value of δ is $\delta_{max} = L/\sqrt{3}$, so for $t \gg \tau_1$, or equivalently for $(\delta_{max} - \delta)/\delta \ll 1$, we expect

$$\frac{d}{dt} \ln \delta = \lambda(\delta) = \frac{1}{\tau_1} \frac{\delta_{max} - \delta}{\delta} \tag{33}$$

which is (16).

Let us remark that in the previous argument for $\lambda(\delta)$ for $\delta \simeq \delta_{max}$ the crucial point is the exponential relaxation to the asymptotic uniform distribution. In a generic deterministic chaotic system it is not possible to prove this property in a rigorous way. Nevertheless one can expect that this request is fulfilled at least in non-pathological cases. In the terminology of chaotic systems the exponential relaxation to asymptotic distribution corresponds to have the second eigenvalue α of the Perron-Frobenius operator inside the unitary circle; now the relaxation time is $\tau_1 = -\ln |\alpha|$ [25].

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FIGURES

FIG. 1.   a) Three realizations of $R^2(t)$ as a function of t built as follows: $R^2(t) = \delta_0^2 \exp(2\gamma t)$ if $R^2(t) < 1$ and $R^2(t) = 2D(t - t_*)$ with $\gamma = 0.08, 0.05, 0.3$ and $\delta_0 = 10^{-7}$, $D = 1.5$. b) $\overline{R^2(t)}$ as function of t averaged on the three realizations shown in figure 1a. The apparent anomalous regime and the diffusive one are shown. c) $\lambda(\delta)$ vs δ, with Lyapunov and diffusive regimes.

FIG. 2.   Lagrangian motion given by the Rayleigh-Bénard convection model with: $A = 0.2$, $B = 0.4$, $\omega = 0.4$, $k = 1.0$, $a = \pi$, the number of realizations is $\mathcal{N} = 2000$ and the series of thresholds is $\delta_n = \delta_0 r^n$ with $\delta_0 = 10^{-4}$ and $r = 1.05$. a)$\lambda(\delta)$ vs δ, the horizontal line indicates the Lyapunov exponent $\lambda = 0.022$, the dashed line is $D\delta^{-2}$ with $D = 0.26$. b)$\overline{R^2(t)}$ as function of t, the line is $2Dt$ with $D = 0.26$.

FIG. 3. $\lambda(\delta)$ vs δ for the same model and parameters of figure 2, but in a closed domain with 6 (crosses) and 12 (diamonds) convective cells. The lines are respectively: (a) Lyapunov regime with $\lambda = 0.017$; (b) diffusive regime with $D = 0.021$; (c) saturation regime with $\delta_{max} = 19.7$; (d) saturation regime with $\delta_{max} = 5.7$.

FIG. 4. $D(\delta)$ vs δ for the modified standard map with $K = 8$, $L = 1000$ and $l = 990$. The series of thresholds is $\delta_n = \delta_0 r^n$ with $\delta_0 = 10^{-4}$ and $r = 2^{1/16}$. The horizontal line indicates the diffusion coefficient in the limit of infinite system, the dashed curve represents the saturation regime.

FIG. 5. $\overline{R^2(t)}$ for the four vortex system with $\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = 1$. The threshold parameter is $r = 1.03$ and $\delta_0 = 10^{-4}$, the dashed line is the power law $\overline{R^2(t)} \sim t^{1.8}$. The number of realizations is $\mathcal{N} = 2000$.

FIG. 6. $\lambda(\delta)$ vs δ for the same model and parameters of figure 5. The horizontal line indicates

the Lyapunov exponent ($\lambda = 0.14$), the dashed curve is the saturation regime with $\delta_{max} = 0.76$.